Problem 13

Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$.

Solution

Combine the two terms in the logarithm's argument.

$$\sum_{n=2}^{\infty} \ln \frac{n^2 - 1}{n^2}$$

Factor the numerator.

$$\sum_{n=2}^{\infty} \ln \frac{(n-1)(n+1)}{n^2}$$

Write the fraction as a product of two terms.

$$\sum_{n=2}^{\infty} \ln \frac{n-1}{n} \cdot \frac{n+1}{n}$$

Write out the first three terms of the sum.

$$\begin{aligned}\sum_{n=2}^{\infty} \ln \frac{n-1}{n} \cdot \frac{n+1}{n} &= \ln \frac{1}{2} \cdot \frac{3}{2} + \ln \frac{2}{3} \cdot \frac{4}{3} + \ln \frac{3}{4} \cdot \frac{5}{4} + \cdots \\ &= \ln \underbrace{\frac{1}{2} \cdot \cancel{\frac{3}{2}}}_{n=2} \cdot \underbrace{\cancel{\frac{2}{3}} \cdot \cancel{\frac{4}{3}}}_{n=3} \cdot \underbrace{\cancel{\frac{3}{4}} \cdot \frac{5}{4}}_{n=4} + \cdots\end{aligned}$$

Every term of n gives us two fractions. The second fraction of n always cancels with the first fraction of $n+1$. Hence, this is a telescoping series. Only the first and last fractions contribute to the value of the sum, which can be determined by calculating a limit.

$$\begin{aligned}\sum_{n=2}^{\infty} \ln \frac{n-1}{n} \cdot \frac{n+1}{n} &= \lim_{n\to\infty} \ln \frac{1}{2} \cdot \frac{n+1}{n} \\ &= \lim_{n\to\infty} \ln \frac{1}{2} \cdot \left(1 + \frac{1}{n}\right) \\ &= \ln \frac{1}{2} \\ &= -\ln 2\end{aligned}$$

Therefore,

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = -\ln 2.$$